discuss the continuity of:

1)* $f(x)=\begin{cases}3x+5 & x>-1\\ -2x & x\le -1\end{cases}$

Solution:

at $x>-1$ the function is continuous (polynomial).

at $x<-1$ the function is continuous (polynomial).

To study the continuity at $x=-1$

$f(-1)=-2\times-1=2$

R.H.L. $=f(-1^+)=$

$\lim\limits_{x\to-1^+}(3x+5)=2$

L.H.L. $=f(-1^-)=\lim\limits_{x\to-1^-}(-2x)=2$

$\therefore f(-1^+)=f(-1^-)=f(-1)$

$\therefore$ The function is continuous at $x=-1$

---

2)*If $f(x)=\begin{cases}4x & x\le -1\\ ax+b & -1<x<3\\ -2x & x\ge 3\end{cases}$

is continuous on R , find the value of each of a and b

Solution:

$\because$ The function is continuous at $x=-1$

$\therefore f(-1)=f(-1^+)=f(-1^-)$

$\therefore 4(-1)=\lim\limits_{x\to-1}(ax+b)$ $\qquad \therefore -4=-a+b$ ①

Also f is constant at $x=3$

$\therefore f(3)=f(3^+)=f(3^-)$ ②

$\therefore -2(3)=\lim\limits_{x\to3}(ax+b)$ $\qquad \therefore -6=3a+b$

Subtract 2 from 1 $\qquad a=\frac{-1}{2}$,

$b=\frac{-9}{2}$

3) Discuss the continuity of the function

$$f(x)=\begin{cases}2x+\sin 2x & -\frac{\pi}{6}<x<0\\ \frac{1}{3}+\cos 2x & 0\le x<\frac{\pi}{6}\end{cases}$$

Solution:

The function is defined on $]\frac{-\pi}{6},\frac{\pi}{6}[$

At $]\frac{-\pi}{6},0[$ the function is continuous

At $]0,\frac{\pi}{6}[$ the function is continuous

Now we will study the continuity at $x=0$

$f(0)=\frac{1}{3}+\cos 0=\frac{1}{3}+1=\frac{4}{3}$

$f(0^+)=\lim\limits_{x\to0}(\frac{1}{3}+\cos 2x)=\frac{4}{3}$

$f(0^-)=\lim\limits_{x\to0}(2x+\sin 2x)=0+0=0$

$\because f(0^+)\ne f(0^-)\Rightarrow$ the function is discontinuous at $x=0$

$\therefore$ the function is continuous on $]\frac{-\pi}{6},\frac{\pi}{6}[-\{0\}$

4)If the function is differentiable at x=1,find the value of each of "a" and "b" $f(x)=\begin{cases}ax^2+bx & x\ge 1\\ 3x-1 & x<1\end{cases}$

$\because$ The fn. is diff. at $x=1$ $\Longrightarrow$

The fn. is cont. at $x=1$

$\because f(1)=f(1^+)=f(1^-)$

$\therefore a+b=2\to$ ①

$\because f'(1^+)=f'(1^-)$

$\therefore 2a+b=3\to$ ②

from ①,②

$a+b=2 \qquad \times -1$

$2a+b=3$

$a=1$ by subs in ① $\Longrightarrow \therefore b=1$

5)Discuss the differentiable of the function where:

$f(x)=x|x|$ at $x=0$

$f(x)=\begin{cases}x^2 & x\ge 0\\ -x^2 & x<0\end{cases}$ ①

$f'(0^+)=\lim\limits_{h\to0^+}\frac{f(0+h)-f(0)}{h}$

$=\lim\limits_{h\to0^+}\frac{0+0+h^2-0}{h}=0$

$f'(0^-)=\lim\limits_{h\to0^-}\frac{f(0+h)-f(0)}{h}$

$=\lim\limits_{h\to0^-}\frac{0+0-h^2-0}{h}=0$

$\because f'(0^+)=f'(0^-)$

$\therefore$ The fn. is diff. at $x=0$

6) Discuss the continuity of the function

$$f(x)=\begin{cases}2x+\sin 2x & -\frac{\pi}{6}<x<0\\ \frac{1}{3}+\cos 2x & 0\le x<\frac{\pi}{6}\end{cases}$$

Solution:

The function is defined on $]\frac{-\pi}{6},\frac{\pi}{6}[$

At $]\frac{-\pi}{6},0[$ the function is continuous

At $]0,\frac{\pi}{6}[$ the function is continuous

Now we will study the continuity at $x=0$

$$f(0)=\frac{1}{3}+\cos 0=\frac{1}{3}+1=\frac{4}{3}$$

$f(0^+)=\lim_{x\to 0^+}(\frac{1}{3}+\cos 2x)=\frac{4}{3}$

$f(0^-)=\lim_{x\to 0^-}(2x+\sin 2x)=0+0=0$

$\because f(0^+)\ne f(0^-)\Rightarrow$

the function is discontinuous at $x=0$

$\therefore$ the function is continuous on $]\frac{-\pi}{6},\frac{\pi}{6}[-\{0\}$

7) $f(x)=\begin{cases}x^2-1, & x>3\\ 2ax+b, & x\le 3\end{cases}$

solution: The fn. is diff. at $x=3$

$\therefore f'(3^+)=f'(3^-)$

$6=6a\Rightarrow a=1$

$\therefore$ The fn. is cont. at $x=3$

$\therefore f(3)=f(3^+)=f(3^-)$

$8=6a+b$

$\therefore b=2$

Find $\frac{dy}{dx}$ of each of the following :

8) If $x=(y^3-1)^5$, prove that:

$$\frac{dy}{dx}=\frac{y^3-1}{15xy^2}$$

solution :

$1=5(y^3-1)^4\cdot 3y^2\frac{dy}{dx}$

$15y^2(y^3-1)^4\frac{dy}{dx}=1$

$\frac{dy}{dx}=\frac{1}{15y^2(y^3-1)^4}\times\frac{(y^3-1)}{(y^3-1)}$

$\frac{dy}{dx}=\frac{y^3-1}{15y^2(y^3-1)^5}$

$\because x=(y^3-1)^5$

$\frac{dy}{dx}=\frac{y^3-1}{15y^2x}$

9) If $y=\sin 2x$ prove that :

$$4\left(\frac{dy}{dx}\right)^2+\left(\frac{d^2y}{dx^2}\right)^2=16$$

solution :

$\frac{dy}{dx}=2\cos 2x$

$\left(\frac{dy}{dx}\right)^2=4\cos^2 2x$

$\frac{d^2y}{dx^2}=-2\sin 2x$

$\left(\frac{d^2y}{dx^2}\right)^2=4\sin^2 2x$

$L.H.S=4(4\cos^2 2x)+4\sin^2 2x$

$=16(\cos^2 x-\sin^2 x)^2+4\sin^2 2x$

$=16(\cos^4 x-2\sin^2 x\cos^2 x+\sin^4 x)$

$+4(4\sin^2 x\cos^2 x)$

$=16(1-2\sin^2 x\cos^2 x)+16\sin^2 x\cos^2 x$

$=16(1-2\sin^2 x\cos^2 x+\sin^2 x\cos^2 x)=16$

10) If $y=3\sin(2x+1)$, prove that :

$$\frac{d^2y}{dx^2}+4y=0$$

solution :

$\frac{dy}{dx}=6\cos(2x+1)$

$\frac{d^2y}{dx^2}=-12\sin(2x+1)$

$L.H.S=-12\sin(2x+1)+12\sin(2x+1)=0$

Geometrical application

11) find the equation of each of the tangent and the normal to the curve $y=\tan x$ at $x=\frac{\pi}{4}$

Solution:

At $x=\frac{\pi}{4}$ ➔ $y=\tan\frac{\pi}{4}=1$

$\therefore$ The point of tangency $=\left(\frac{\pi}{4},1\right)=\left(\frac{11}{14},1\right)$

$\because y=\tan x$ $\therefore \frac{dy}{dx}=\sec^2 x$

The slope of the tangent at $\left(\frac{\pi}{4},1\right)=\left(\frac{dy}{dx}\right)_{(\frac{\pi}{4},1)}=\sec^2\frac{\pi}{4}=\sec^2 45=2$

So the slope of the normal $=\frac{-1}{2}$

The equation of the tangent

$\frac{y-y_1}{x-x_1}=m$

$\therefore \frac{y-1}{x-\frac{11}{14}}=2$ →

$2x-\frac{11}{7}=y-1$ mult. by 7

$\therefore \boxed{14x-11=7y-7}$ →

$14x-7y-4=0$

equation of the tangent

The equation of the normal ➔

$\frac{y-1}{x-\frac{11}{14}}=\frac{-1}{2}$

$\therefore \boxed{14x+28y-39=0}$ →

equation of the normal

12) prove that the equation of the tangent to the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1,y_1) on the curve is

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$$

Solution:

$\because \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Differentiating with respect to x

$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx}=\frac{-2x}{a^2}\times\frac{b^2}{2y}=\frac{-xb^2}{ya^2}$ → slope of the tangent at any point

$\therefore$ The slope of the tangent at $(x_1,y_1)=\frac{-x_1}{y_1}\times\frac{b^2}{a^2}$

∴ The equation of the tangent at (x_1, y_1)

$$\frac{y-y_1}{x-x_1}=\frac{-x_1}{y_1}\times\frac{b^2}{a^2}$$

$$\therefore y_1 a^2 (y-y_1) = -x_1 b^2 (x-x_1) \rightarrow$$

$$\frac{y_1}{b^2}(y-y_1)=\frac{-x_1}{a^2}(x-x_1)$$

$$\therefore \frac{y_1 y}{b^2}-\frac{y_1^2}{b^2}=\frac{-x_1 x}{a^2}+\frac{x_1^2}{a^2}$$

$$\therefore \frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

∵ (x_1, y_1) satisfy the equation of the curve

$$\therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

$$\therefore \frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=1 \rightarrow$$

equation of the tangent

Related time rates

13) A particle moves along the curve $y^2+x^2-5x+3y-6=0$, if the rate of change of its x-coordinate with respect to time as it passes through the point (1,2) equals 3. Find the rate of change of its y-coordinate with respect to time at the same time.

Solution:

∵ $y^2+x^2-5x+3y-6=0$

Differentiating with respect to t

$$\therefore 2y\frac{dy}{dt}+2x\frac{dx}{dt}-5\frac{dx}{dt}+3\frac{dy}{dt}=0$$

$$\therefore (2y+3)\frac{dy}{dt}+(2x-5)\frac{dx}{dt}=0$$

at $(1,2) \rightarrow \frac{dx}{dt}=3$

$$\therefore (4+3)\frac{dy}{dt}+(2-5)(3)=0$$

$$\therefore \frac{dy}{dt}=\frac{9}{7}$$

14) If the area of a circular disk is increasing at rate of 0.2 cm^2/sec. find the rate of increases of its radius when its radius is 7 cm $\left(\pi=\frac{22}{7}\right)$.

Solution:

Let A is the area of the disk

$A=\pi r^2$

$$\therefore \frac{dA}{dt}=2\pi r\frac{dr}{dt}$$

At $r=7 \rightarrow \frac{dA}{dt}=0.2$

$\therefore 0.2=2\times$

$$\frac{22}{7}\times 7\times\frac{dr}{dt} \rightarrow \frac{dr}{dt}=\frac{1}{220}\ cm/sec$$

15) A rectangular lamina, the length of its base is greater than its height by 20 cm, shrinks by cooling so that the difference between the length of its base and its height remains the same. If the base of the lamina shrinks at the rate 0.025 cm/sec. when its height is 80 cm, find the rate of change of its area at that instant.

Solution:

Area = A = x(x−20)

$\therefore A=x^2-20x$

$$\therefore \frac{dA}{dt}=2x\frac{dx}{dt}-20\frac{dx}{dt}$$

$\because \frac{dx}{dt}=-0.025$ (shrinking)

at x = 80 + 20 = 100

$$\therefore \frac{dA}{dt}=2\times100\times(-0.025)-20\times(-0.025)$$

$$=-4.5$$

16) A man of height 1.5 m walks with velocity 2 m/sec away from a light pole having a lamp hung at its top. If the height of the pole from the ground is 3 m.

Find: 1) The rate of change of the length of the man's shadow.

2) The velocity of the end of the man's shadow.

3) The rate of change of the distance between the man's head and the lamp.

When the man is at distance 2 m from the pole.

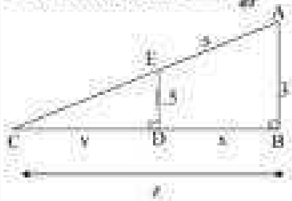
Solution:

Let the distance between the man and the pole is (x)

Let the length of the man's shadow is (y)

let the distance between the man's head and the lamp is (s)

∵ The velocity of the man $=\frac{dx}{dt}=2$



From symmetry of Δ ABC, Δ EDC

$$\frac{1.5}{3}=\frac{y}{y+x} \qquad \therefore y=x$$

$$\therefore \frac{dy}{dt}=\frac{dx}{dt} \qquad \therefore \frac{dy}{dt}=2\ m/sec \rightarrow$$

Rate of change of man's shadow

Let z is the distance of the end of the man's shadow

$\therefore z = x + y$

$\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = 2+2 = 4$ m/sec

➔ velocity of the end of the man's shadow

$\because$ D is the mid-point of $\overline{BC}$, $\overline{DE} // \overline{AB}$

$\therefore AE = EC = S$

$\therefore S^2 = (1.5)^2 + y^2 = \frac{9}{4} + x^2$

$\therefore 2S\frac{dS}{dt} = 2x\frac{dx}{dt}$

at $x = 2$ $\quad \frac{dx}{dt} = 2 \quad \therefore S^2 = \frac{9}{4} + 4 = \frac{25}{4}$

$\therefore 2 \times \frac{5}{2} \times \frac{dS}{dt} = 2 \times 2 \times 2 \Rightarrow \quad \frac{dS}{dt} = \frac{8}{5}$ cm / sec.

<u>17):</u> prove that the function f(x)=|x−1| has a critical point at x=1 then show that it is local min. value of the function.

<u>Solution:</u>

$f(x) = |x-1| = \begin{cases} x-1 & x \geq 1 \\ -x+1 & x < 1 \end{cases}$

$f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0^+} \frac{(1+h-1) - (1-1)}{h} = \boxed{1}$

$f'(1^-) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0^-} \frac{(-1-h+1) - (1-1)}{h} = \boxed{-1}$

$\therefore f'(1^+) \neq f'(1^-)$

$\therefore f'(1)$ doesn't exist and f(1) exist

$\therefore$ x=1 is a critical point

At $x \in]-\infty, 1[$ ➔ $f'(x) = -1 < 0$ dec.

At $x \in]1, \infty[$ ➔ $f'(x) = 1 > 0$ inc.

$\therefore$ at x=1 f(x) has local min. value.

<u>18)</u> find the values of a, b given that the function $f(x) = ax + \frac{b}{x}$ has a critical point at x=3, and f(3)=2, then determine the type of the point (3,2) whether it is a point of local max. or min. value.

<u>Solution:</u>

$\because f(3) = 2 \Rightarrow \quad 3a + \frac{b}{3} = 2 \qquad$ mult. by 3

$9a + b = 6 \qquad (1)$

$f'(x) = a - \frac{b}{x^2}$

$\because$ x=3 is a critical point ➔ $f'(3)=0$

$\therefore a - \frac{b}{9} = 0 \qquad$ mult. by 9

$9a - b = 0 \qquad (2)$

By solving (1) and (2) ➔ $a = \frac{1}{3}$,

$b = 3$

$f''(x) = \frac{2b}{x^3}$

$f''(3) = \frac{2b}{27} = \frac{2 \times 3}{27} = \frac{2}{9} > 0$

$\therefore$ f has local min. value at x=3

<u>19)</u> Find the local max. and min. of f(x)=sin x cos x in [0, π]

<u>Solution:</u>

$f(x) = \sin x \cos x = \frac{1}{2}\sin 2x$

$\therefore f'(x) = \cos 2x$

$f'(x) = 0$ when

$2x = \frac{\pi}{2} \quad$ or $\quad 2x = \frac{3\pi}{2}$

$\therefore x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$ (Critical points

$f\left(\frac{\pi}{4}\right) = \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}$

$f\left(\frac{3\pi}{4}\right) = \frac{1}{2}\sin\frac{3\pi}{2} = \frac{-1}{2}$

$f(0) = \frac{1}{2}\sin 0 = 0$

$f(\pi) = \frac{1}{2}\sin \pi = 0$

$\therefore$ f has max. value at $x = \frac{\pi}{4}$,

$f\left(\frac{\pi}{4}\right) = \frac{1}{2}$

F has min. value at $x = \frac{3\pi}{4}$

$f\left(\frac{3\pi}{4}\right) = \frac{-1}{2}$

---

20) Determine the interval over which the curve of f(x) is convex upwards or convex downwards and the points of inflection.

$f(x) = x^4 - 6x^3 + 12x^2 - 8x$

$f'(x) = 4x^3 - 18x^2 + 24x - 8$

$f''(x) = 12x^2 - 36x + 24$

$f''(x) = 0 \quad$ when $\quad 12x^2 - 36x + 24 = 0 \qquad \div 12$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$x = 1 \quad , \quad x = 2$

at $]-\infty, 1[\Rightarrow f''(x) > 0$

$\therefore$ the curve is convex downwards

at $]1, 2[\Rightarrow f''(x) < 0$

$\therefore$ the curve is convex upwards

at $]2, \infty[\Rightarrow f''(x) > 0$

the curve is convex downwards

there are inflection points at $x = 1$, $x = 2$

$f(1) = (1)^4 - 6(1)^3 + 12(1)^2 - 8(1) = -1$

$\therefore (1, -1)$ is an inf. Pt

$f(2)=(2)^3-6(2)^2+12(2)^2-8(2)=-1$

$\therefore (2,0)$ is an inf. Pn

---

21)If the curve $y=x^3+ax^2+bx$ has an inflection point a at $(1,-3)$, find the value of a , b

SOL:

$y'=3x^2+2ax+b$

$y''=6x+2a$

$\because (1,-3)$ is an inf. pn

$\therefore y''=0$ at $x=1$

$\therefore 6(1)+2a=0 \Rightarrow \boxed{a=-3}$

$\because f(1)=-3 \Rightarrow (1)^3+a(1)^2+b(1)=-3$

$1-3+b=-3 \Rightarrow \boxed{b=-1}$

---

22)Find the equation of inflection tangent to the curve

$y=x^3-6x^2+x-2$

SOL.

$y'=3x^2-12x+1$, $y''=6x-12$

$y''=0$ when $6x-12=0 \Rightarrow x=2$

at $]-\infty,2] \Rightarrow f''(x)<0$

$\therefore$ the curve is convex upwards.

at $[2,\infty[\Rightarrow f''(x)>0$

$\therefore$ the curve is convex downwards.

$f(2)=(2)^3-6(2)^2+2-2=-16$

$(2,-16)$ is a pn of inflection.

To find the equation of the tangent at $(2,-16)$

the slope of the tangent $=$

$[y']_{(2,-16)}=3(2)^2-12(2)+1=-11$

the eqn. of the tangent $\frac{y-y_1}{x-x_1}=m$

$\frac{y+16}{x-2}=\frac{-11}{1} \Rightarrow y+16=-11x+22$

$11x+y-6=0$

Sketching the fn

23)Graph the fn

$f(x)=x^3-3x^2+4$

SOL:

$f'(x)=3x^2-6x$

$f''(x)=6x-6$

$f'(x)=0$ when $3x^2-6x=0$ $\div 3$

$x^2-2x=0$

$x(x-2)=0$

$x=0$, $x=2$ critical points

$f''(0)=-6<0$ L. max. value

$f(0)=4$ $\therefore$ the fn has L. max. value at $(0,4)$

$f''(2)=6>0$ L. min. value

$f(2)=(2)^3-3(2)^2+4=0$

$\therefore f$ has L. min. value at $(2,0)$

$f''(x)=0$ when $6x-6=0$, $x=1$

at $]-\infty,1] \Rightarrow f''(x)<0$

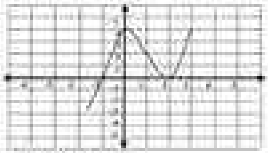
$\therefore$ the curve is convex upwards

at $[1,\infty[\Rightarrow f''(x)>0$

$\therefore$ the curve is convex downwards

$f(1)=(1)^3-3(1)^2+4=2$

$\therefore (1,2)$ is a point of inflection.



24)*Graph the fn :

$f(x)=(2-x)(x-1)^2$

$=(2-x)(x^2+2x+1)$

$=2x^2+4x+2-x^3-2x^2-x$

$=-x^3+3x+2$

$f'(x)=-3x^2+3$

$f''(x)=-6x$

$f'(x)=0$ when $-3x^2+3=0$

$x^2=1 \Rightarrow x=\pm 1$ critical points

$f''(1)=-6<0 \rightarrow$ L. max. pn

$f(1)=-(1)^3+3(1)+2=4$

f has L. max. value at $(1,4)$

$f''(-1)=6>0 \rightarrow$ L. min. value

$f(-1)=-(-1)^3+3(-1)+2=0$

$\therefore f$ has L. min. value at $(-1,0)$

$f''(x)=0$ when $x=0$

at $]-\infty,0] \Rightarrow f''(x)>0$

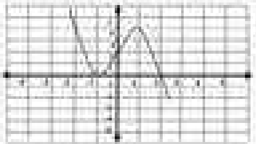
$\therefore$ the curve is convex downwards

at $[0,\infty[\Rightarrow f''(x)<0$

the curve is convex upwards

$f(0)=2$ $(0,2)$ is pn of inflection

X	-2	-1	0	1	2
$f(x)$	4	0	2	4	0



Applications on max. & min. values

*25) A rectangle is to be constructed using a wire of length 20 cm. find its dimensions if its area is to be maximum.

Solution

Let the length of one dimensions $= x$

The length of the other dimension $=$ $10 - x$

where $x \in]0, 10[$

The area of the rectangle

$A(x) = x(10 - x)$

$A(x) = 10x - x^2$

$A'(x) = 10 - 2x$

$A''(x) = -2$

$A'(x) = 0$ when $10 - 2x = 0 \Rightarrow x = 5$

$A''(5) = -2 < 0$

$\therefore$ the area is to be max. when one of the two dimensions $= 5$

The other dimension $= 10 - 5 = 5$ (square

$A(5) = 5(5) = 25 \rightarrow$ max. area

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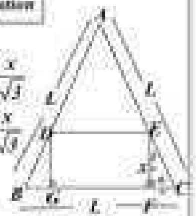
26) A rectangle is to be drawn inside an equilateral triangle of side L so that one side of the rectangle lies on the base of the triangle and the vertices of the opposite sideof the rectangle lie on the two other sides of the triangle. Find the dimensions of the rectangle if its area is to be maximum.

Solution



Let $EF = x$

$CF = x \tan 30 = \frac{x}{\sqrt{3}}$

Similarly $BG = \frac{x}{\sqrt{3}}$

$\therefore GF = L - \frac{2x}{\sqrt{3}}$

The area of the rectangle

$A(x) = x(L - \frac{2x}{\sqrt{3}})$

$\therefore A(x) = xL - \frac{2}{\sqrt{3}}x^2$

$A'(x) = L - \frac{4}{\sqrt{3}}x$

$A''(x) = -\frac{4}{\sqrt{3}}$

$A'(x) = 0$ when

$x = \frac{\sqrt{3}}{4}L$ critical point

$A''(\frac{\sqrt{3}}{4}L) = -\frac{4}{3} < 0$

the area is max. when one of the two dimensions is $\frac{\sqrt{3}}{4}L$

$\therefore$ the other dimension

$L - \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{4}L\right) = \frac{1}{2}L$

$\therefore$ the max. area $A\left(\frac{\sqrt{3}}{4}L\right) = \frac{\sqrt{3}}{4}L \times \frac{1}{2}L = \frac{\sqrt{3}}{8}L^2$

*27) A factory producing electric appliances profits L.E 30 in every appliance, if it produces 50 appliances monthly, when the production increased than that, the profit in the appliance decreases by 50 piaster for every extra appliance produced. Find the number of appliances produced monthly if the profit is to be max.

Let the number of appliances increase by x i.e the monthly production $= 50 + x$

the profit will decrease $0.5x$

the profit of each appliance $= 30 - 0.5x$

$\therefore$ total profit

$P(x) = (50 + x)(30 - 0.5x)$

$= 1500 + 5x - \frac{1}{2}x^2$

$\therefore P'(x) = 5 - x$

$P''(x) = -1$

$P'(x) = 0$ when $5 - x = 0 \Rightarrow x = 5$

$P''(5) = -1 < 0$ max. value

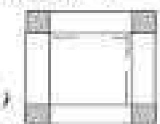
the n^0 of appliances produced monthly to give max. profit is 55

28) A box with an open top is to be constructed by cutting equal squares from the corners of a square thin metallic lamina with sides 10 cm long and turning up the sides. Find the length of the side of the removed square if the volume of the box is to be max.

Solution

Let the side of the remove square $= x$

$\therefore x \in]0, 5[$



The volume of the box

$= x(10 - 2x)(10 - 2x)$

$\therefore V(x) = 100x - 40x^2 + 4x^3$

$\therefore V'(x) = 100 - 80x + 12x^2$

$V''(x) = -80 + 24x$

$V'(x) = 0$ when $100 - 80x + 12x^2 = 0$ $\div 4$

$3x^2 - 20x + 25 = 0$

$(3x - 5)(x - 5) = 0$

$\therefore x = \frac{5}{3}$, $x = 5$

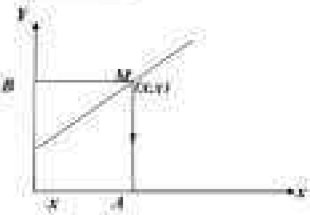
$f''(\frac{5}{3}) = -80 + 40 < 0$ max. value

$f''(5) = -80 + 120 > 0$ min. value

∴ the volume of the box is to be max. if the side length of the removed square is $\frac{5}{3}$ cm, $V(\frac{5}{3}) = \frac{2000}{27}$

29) A point $M = (x, y)$ moves on the st. line whose equation is $y = 3(8 - x)$, where $x > 0$, $y > 0$. If the point A is the projection of M on the axis $\overleftrightarrow{X'OX}$ and the point B is the projection of M on the axis $\overleftrightarrow{Y'OY}$, and S is the surface area of the rectangle $OAMB$ (where O is the origin). Find S in terms of x, then find the value of x at which S is max.

Solution



$S = xy$

$\because y = 3(8 - x)$

$S = 3x(8 - x)$

$S = 24x - 3x^2$

$S' = 24 - 6x$

$S'' = -6$

$S' = 0$ when $24 - 6x = 0 \Rightarrow x = 4$

$S''(4) = -6 < 0$ the area of the rectangle is to be max. when $x = 4$

$S(4) = 12(8 - 4) = 48$ unit squar

30) $ABCD$ is a rectangle, in which $AB = 6$ cm, $BC = 8$ cm, $\overrightarrow{AQ}$ is drawn to intersect $\overline{CD}$ at P and $\overrightarrow{BC}$ at Q. Find the tangent of the angle BAQ when the sum of the surface areas of the triangles ADP and PCQ is to be minimum. Solution

Let $DP = x$ cm … $PC = 6 - x$

$\triangle ADP \sim \triangle QCP$

$\frac{CQ}{8} = \frac{6-x}{x} \Rightarrow CQ = 8(\frac{6}{x} - 1) = \frac{48}{x} - 8$

the sum of areas of

$\triangle ADP + \triangle QCP$

$S(x) = \frac{1}{2}(8)x + \frac{1}{2}(6 - x)(\frac{48}{x} - 8)$

$\therefore S(x) = 4x + \frac{144}{x} - 48 + 4x$

$= 8x + \frac{144}{x} - 48$

$S'(x) = 8 - \frac{144}{x^2}$

$S''(x) = \frac{288}{x^3}$

$S'(x) = 0$ when $8 - \frac{144}{x^2} = 0$

$\therefore x^2 = 18 \Rightarrow x = 3\sqrt{2}$

$S''(3\sqrt{2}) = \frac{288}{(3\sqrt{2})^3} > 0$

min. value

$\tan\theta = \frac{8}{x} = \frac{8}{3\sqrt{2}} = \frac{4}{3}\sqrt{2}$

31) The sum of the circumference of a circle and the perimeter of a square is 80 cm. prove that when the sum of the areas of both figures is minimum, then the diameter of the circle equals to the side length of the square. Solution

Let the radius of the circle = x

Let the side length = y

$2\pi x + 4y = 80$

$\therefore y = 20 - \frac{1}{2}\pi x$ ——— (1)

Let the sum of areas of the square and the circle is S

$S = \pi x^2 + y^2$

subst. from (1)

$\therefore S(x) = \pi x^2 + (20 - \frac{1}{2}\pi x)^2$

$S'(x) = 2\pi x + 2(20 - \frac{1}{2}\pi x)(-\frac{1}{2}\pi)$

$= 2\pi x - 20\pi + \frac{1}{2}\pi^2 x$

$S''(x) = 2\pi + \frac{1}{2}\pi^2$

$S'(x) = 0$ when $2\pi x - 20\pi + \frac{1}{2}\pi^2 x = 0$ $\div\pi$

$\therefore 2x + \frac{1}{2}\pi x = 20$ $\times 2$

$x(4 + \pi) = 40 \Rightarrow x = \frac{40}{4 + \pi} \rightarrow$ Radius

$S''(\frac{40}{4+\pi}) = 2\pi + \frac{1}{2}\pi^2 > 0$ min value

∴ the diameter of the circle $= 2x = \frac{80}{4 + \pi}$

The side length of the square = y

$= 20 - \frac{1}{2}\pi \frac{40}{4+\pi}$

$\therefore y = \frac{20(4+\pi) - 20\pi}{4+\pi}$

$$\frac{80}{4+\pi} = \text{diameter}$$

32) Determine the following integrals

1) $\int (\sqrt{x} - 1)^2 dx = \int (x - 2\sqrt{x} + 1) dx$
$= \int (x - 2x^{\frac{1}{2}} + 1) dx$
$= \frac{1}{2}x^2 - \frac{4}{3}x^{\frac{3}{2}} + x + c$

2) $\int (x^2 + 2)^3 dx = \int (x^6 + c_1 x^4 (2) + c_2 x^2 (2)^2 + 2^3) dx$
$= \int (x^6 + 6x^4 + 12x^2 + 8) dx$
$= \frac{1}{7}x^7 + \frac{6}{5}x^5 + 4x^3 + 8x + c$

3) $\int (7x - 3)^8 dx = \frac{1}{7 \times 9}(7x - 3)^9 + c$
$= \frac{1}{63}(7x - 3)^9 + c$

4) $\int \frac{5}{(2x-1)^5} dx = \int 5(2x - 1)^{-5} dx$
$= 5 \times \frac{(2x-1)^{-4}}{2 \times -4} + c$
$= -\frac{5}{8}(2x - 4)^{-4} + c$

5) $\int (\sec^2 x + \cos x) dx = \tan x + \sin x + c$

6) $\int \cos(2x + 3) dx = \frac{1}{2}\sin(2x + 3) + c$

7) $\int \sin\frac{x}{2} dx = \int \sin\frac{1}{2}x\, dx = -2\cos\frac{1}{2}x + c$

8) $\int (\sin x + \cos x)^2 dx$
$= \int (\sin^2 x + 2\sin x \cos x + \cos^2 x) dx$
$= \int (1 + \sin 2x) dx = x - \frac{1}{2}\cos 2x + c$

9) $\int (1 + \cos x)^2 dx$
$= \int (1 + 2\cos x + \cos^2 x) dx$
$= \int (1 + 2\cos x + \frac{1}{2}(1 + \cos 2x)) dx$

$\because \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$= \int (\frac{3}{2} + 2\cos x + \frac{1}{2}\cos 2x) dx$

$\because \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= \frac{3}{2}x + 2\sin x + \frac{1}{4}\sin 2x + c$

10) $\int x\sqrt{\frac{6}{x} + \frac{5}{x^2}}\, dx = \int \sqrt{x^2\left(\frac{6}{x} + \frac{5}{x^2}\right)}\, dx$
$= \int (6x + 5)^{\frac{1}{2}} dx$
$= \frac{(6x+5)^{\frac{3}{2}}}{6 \times \frac{3}{2}} + c = \frac{1}{9}(6x + 5)^{\frac{3}{2}} + c$

11) $\int \cos(\frac{x}{2} + 2) dx = 2\sin(\frac{x}{2} + 2) + c$

12) $\int x\sqrt{x + 1}\, dx$
$= \int [(x + 1) - 1]\sqrt{x + 1}\, dx$
$= \int (x + 1)\sqrt{x + 1}\, dx - \int \sqrt{x + 1}\, dx$
$= \int (x + 1)^{\frac{3}{2}} dx - \int (x + 1)^{\frac{1}{2}} dx$
$= \frac{2}{5}(x + 1)^{\frac{5}{2}} - \frac{2}{3}(x + 1)^{\frac{3}{2}} + c$

13) $\int \frac{x - 1}{(2x + 1)^3} dx = \frac{1}{2}\int \frac{2x - 2}{(2x + 1)^3} dx$
$= \frac{1}{2}\int \frac{(2x + 1) - 3}{(2x + 1)^3} dx$
$= \frac{1}{2}\int \frac{(2x + 1)}{(2x + 1)^3} dx - \frac{3}{2}\int \frac{1}{(2x + 1)^3} dx$
$= \frac{1}{2}\int (2x + 1)^{-2} dx - \frac{3}{2}\int (2x + 1)^{-3} dx$
$= \frac{1}{2} \frac{(2x + 1)^{-1}}{-1 \times 2} - \frac{3}{2} \times \frac{(2x + 1)^{-2}}{2 \times -2} + c$
$= -\frac{1}{4}(2x + 1)^{-1} + \frac{3}{8}(2x + 1)^{-2} + c$

Applications on integral

14) Find the eqⁿ. of the curve passing through the point (1, 2) and the slope of its tangent at any point (x, y) on it equals $3x^2$

■ Solution

$\frac{dy}{dx} = 3x^2 \quad \int 3x^2 dx$

$\therefore y = x^3 + c$

(1, 2) lies on the curve

$2 = (1)^3 + c \Rightarrow c = 1$

$y = x^3 + 1$

15) Find the equation of the curve whose slope at any point (x, y) is (3x + 1)(x − 1) and passing through the pᵗ (1, 2)

■ Solution

$\frac{dy}{dx} = (3x + 1)(x - 1)$

$\therefore \frac{dy}{dx} = 3x^2 - 2x - 1$

$\therefore y = \int (3x^2 - 2x - 1) dx$
$= x^3 - x^2 - x + c$

the curve passing through (1, 2)

$\therefore 2 = (1)^3 - (1)^2 - (1) + c \Rightarrow c = 3$

$\therefore y = x^3 - x^2 - x + 3$

$\therefore \frac{dy}{dx} = 2x$

$\therefore y = \int 2x\, dx$

$\therefore y = x^2 + c$, c const.

16) If the rate of change of the slope of the tangent to the curve is 6(2x − 3). Find the eqⁿ. of the curve given that it passes through the two points (1, 5), (0, 4)

■ Solution

$\frac{d^2y}{dx^2} = 6(2x - 3)$

$\therefore \frac{dy}{dx} = \int (12x - 18) dx$

$\therefore \frac{dy}{dx} = 6x^2 - 18x + c_1$

$\therefore \quad y = \int (6x^2 - 18x + c_1)\, dx$

$= 2x^3 - 9x^2 + c_1 x + c_2$

$(1, 5)$ lies on the curve

$\therefore 5 = 2(1)^3 - 9(1)^2 + c_1(1) + c_2$

$\therefore c_1 + c_2 = 12$ ——— (1)

$(0, 4)$ lies on the curve

$\therefore \quad 4 = 0 - 0 + 0 + c_2 \Rightarrow c_2 = 4$

Subst. in $(1) \Rightarrow c_1 = 8$

$\therefore \quad y = 2x^3 - 9x^2 + 8x + 4$

17) If the slope of the tangent to the curve at any point is $3(x-3)(x-5)$, $y = 4$ is local maximum value. Find the L. min. value of this curve.

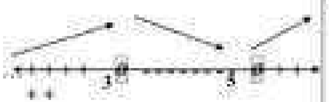
■ Solution

$\frac{dy}{dx} = 3(x-3)(x-5)$

At the critical points $\frac{dy}{dx} = 0$

$\therefore (x-3)(x-5) = 0$

$\therefore x = 3 \quad , x = 5$



at $]-\infty, 3[\Rightarrow \frac{dy}{dx} > 0$ inc.

at $]3, 5[\Rightarrow \frac{dy}{dx} < 0$ dec.

at $]5, \infty[\Rightarrow \frac{dy}{dx} > 0$ inc.

at $x = 3 \to$ L. max. value

at $x = 5 \to$ L. min value

$(3, 4)$ is the pt of L. max. value on the curve

The eqn of the curve is

$y = \int 3(x-3)(x-3)\, dx$

$\therefore \quad y = \int (3x^2 - 24x + 45)\, dx$

$= x^3 - 12x^2 + 45x + c$

the curve passes through $(3, 4)$

$\therefore 4 = 27 - 108 + 135 + c \Rightarrow c = -50$

$\therefore y = x^3 - 12x^2 + 45x - 50$

this curve has L. min. value at $x = 5$

$[y]_{x=5} = (5^3) - 12(5)^2 + 45(5) - 50 = 0$

The L. min value $\quad y = 0$

18) If the slope of the tangent to the curve at any point (x, y) is $\frac{3x}{y}$ find the eqn of the curve given that it passes through $(3, 5)$.

■ Solution

$\frac{dy}{dx} = \frac{3x}{y} \quad \therefore y \frac{dy}{dx} = 3x$

$\therefore \int y \frac{dy}{dx}\, dx = \int 3x\, dx$

$\frac{1}{2} y^2 = \frac{3}{2} x^2 + c$

$(3, 5)$ lies on the curve

$\therefore \frac{1}{2}(25) = \frac{3}{2}(9) + c \Rightarrow c = -1$

$\therefore \frac{1}{2} y^2 = \frac{3}{2} x^2 - 1$

$\therefore y^2 = 3x^2 - 2 \quad$ eqn. of the curve

18) Find the eqn of the curve $y = f(x)$ if $\frac{d^2y}{dx^2} = 18 \sin 3x$ and the equation of the tangent at $(0, 5)$ is $x + y - 5 = 0$

■ Solution

$\frac{d^2y}{dx^2} = 18 \sin 3x$

$\therefore \quad \frac{dy}{dx} = \int 18 \sin 3x\, dx$

$\frac{dy}{dx} = -6 \cos 3x + c \quad (1) \to$

slope of the tangent at any point

the slope of the given tangent $= \frac{-\text{coeff of } x}{\text{coeff of } y} = -1$ ___ (2)

at $(0, 5)$

from (1), (2)

$\left[\frac{dy}{dx}\right]_{x=0} = -1$

$-6 \cos 0 + c = -1$

$-6(1) + c = -1 \quad \therefore c = 5$

$\therefore \frac{dy}{dx} = 5 - 6 \cos 3x$

$y = \int (5 - 6 \cos 3x)\, dx$

$y = 5x - 2 \sin 3x + c'$

$\because (0, 5)$ lies on the curve

$\therefore 5 = 0 - 0 + c' \Rightarrow c' = 5$

$\therefore \quad y = 5x - 2 \sin 3x + 5 \to$ eqn of the curve

19) The slope of the normal to a curve at any point (x, y) on it is $\sqrt{3-2x}$ find its equation given that it passes through the point $(1, 3)$.

Solution

the slope of the normal $= \sqrt{3-2x}$

the slope of the tangent $= -\frac{1}{\sqrt{3-2x}}$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{3-2x}}$

$\therefore \quad y = -\int \frac{1}{\sqrt{3-2x}}\, dx$

$= -\int (3-2x)^{-\frac{1}{2}} dx$

$y = -\frac{(3-2x)^{\frac{1}{2}}}{-2 \times \frac{1}{2}} + c$

$y = \sqrt{3-2x} + c$

$(1, 3)$ lies on the curve

$\therefore 3 = \sqrt{3-2(1)} + c \Rightarrow c = 2$

$\therefore \quad y = \sqrt{3-2x} + 2$